Example Let $\lim_n x_n = 3$. Show by $\varepsilon-N$ that

$$\lim_n \frac{x_n^3+1}{x_n-2} = 28.$$

Analysis (How to think):

I. $$\left|\frac{x_n^3+1}{x_n-2} - 28\right| = \left|\frac{x_n^3-28x_n+57}{x_n-2}\right| = \frac{|(x_n-3)(x_n^2+3x_n-19)|}{|x_n-2|} \quad (\#)$$

Thus, when x_n near 3 (say $x_n \in (3-\varepsilon_0, 3+\varepsilon_0)$), we must ensure that $|x_n-2|$ in the denominator is greater than a positive constant c. Since

$$3-\varepsilon_0 < x_n < 3+\varepsilon_0 \Rightarrow 1-\varepsilon_0 < x_n-2 < 1+\varepsilon_0.$$

Our natural choice for c is $1-\varepsilon_0$ and hence ε_0 should a positive number strictly less than 1 (so that $c := 1-\varepsilon_0 > 0$)

For simplicity, we take $\varepsilon_0 = \frac{1}{2}$. Then $\forall x_n \in V_{\varepsilon_0}(3) = (3-\frac{1}{2}, 3+\frac{1}{2})$ one has $|x_n-2| > \frac{1}{2}$ and $|x_n| < 3+\varepsilon_0 < 4$ and $|x_n^2+3x_n-19| < 4^2+3\times4+19 < 50$

(since $x_n^2+3x_n-19$ maybe negative, so advisable to estimate with absolute values for upper bound). Thus

$$\begin{aligned}(\#) \le \frac{|x_n-3|\cdot 50}{1/2} &= 100|x_n-3| && \text{if } x_n \in V_{1/2}(3) \\ &< 100\,\varepsilon' && \text{if also } |x_n-3| < \varepsilon' \\ &\le \varepsilon && \text{if } \varepsilon' \le \frac{\varepsilon}{100}\end{aligned}$$

Therefore

We are led to consider $\varepsilon' = \min\left\{\frac{1}{2}, \frac{\varepsilon}{100}\right\}$.

II. The numerator on the RHS of (#) has two factors one of which is unfailingly to be "as small as you like" (as $\lim x_n = 3$) factor $|x_n - 3|$ (unless you make a mistake in the computation!) while the other factor is bounded.

The formal solution can now be as below:

Let $\varepsilon > 0$. Take $\varepsilon' > 0$ such that

$$\varepsilon' := \min\left\{\frac{1}{2}, \frac{\varepsilon}{100}\right\}. \qquad (*)$$

Since $\lim x_n = 3$, $\exists N \in \mathbb{N}$ s.t. $|x_n - 3| < \varepsilon'$ $\forall n \geq N$

Let $n \geq N$. Then $|x_n - 3| < \varepsilon' \leq 1/2$ so

$3 - 1/2 < x_n < 3 + 1/2$ and so $1/2 < |x_n - 2|$ and $x_n < 4$. It remains to show that

$$\left|\frac{x_n^3 + 1}{x_n - 2} - 28\right| < \varepsilon \qquad (?)$$

To check this, note that

$$\left|\frac{x_n^3+1}{x_n-2} - 28\right| = \frac{|x_n - 3|\,|x_n^2 + 3x_n - 19|}{|x_n - 2|} \leq \frac{|x_n - 3| \cdot (|x_n^2| + 3|x_n| + 19)}{|x_n - 2|}$$

$$< \frac{\varepsilon' \cdot (4^2 + 3 \times 4 + 19)}{1/2} = 100\varepsilon' \leq \varepsilon,$$

as was wished to show in (?). QED